ON THE DANILOV-GIZATULLIN ISOMORPHISM THEOREM

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ABSTRACT. A *Danilov-Gizatullin surface* is a normal affine surface $V = \Sigma_d \setminus S$ which is a complement to an ample section S in a Hirzebruch surface Σ_d. By a surprising result due to Danilov and Gizatullin [DaGi] V depends only on $n = S^2$ and neither on d nor on S. In this note we provide a new and simple proof of this Isomorphism Theorem.

1. The Danilov-Gizatullin theorem

By definition, a Danilov-Gizatullin surface is the complement $V = \Sigma_d \setminus S$ of an ample section S in a Hirzebruch surface Σ_d, $d \geq 0$. In particular $n := S^2 > d$. The purpose of this note is to give a short proof of the following result of Danilov and Gizatullin [DaGi, Theorem 5.8.1].

Theorem 1.1. *The isomorphism type of $V_n = \Sigma_d \setminus S$ only depends on n. In particular, it neither depends on d nor on the choice of the section S.*

For other proofs we refer the reader to [DaGi] and [CNR, Corollary 4.8]. In the forthcoming paper [FKZ_2, Theorem 1.0.5] we extend the Isomorphism Theorem 1.1 to a larger class of affine surfaces. However, the proof of this latter result is much harder.

2. Proof of the Danilov-Gizatullin theorem

2.1. **Extended divisors of Danilov-Gizatullin surfaces.** Let as before $V = \Sigma_d \setminus S$ be a Danilov-Gizatullin surface, where S is an ample section in a Hirzebruch surface Σ_d, $d \geq 0$ with $n := S^2 > d$. Picking a point, say, $A \in S$ and performing a sequence of n blowups at A and its infinitesimally near points on S leads to a new SNC completion $(\bar{V}, D)$ of V. The new boundary $D = C_0 + C_1 + \ldots + C_n$ forms a *zigzag* i.e., a linear chain of rational curves with weights $C_0^2 = 0$, $C_1^2 = -1$ and $C_i^2 = -2$ for $i = 2, \ldots, n$. Here $C_0 \cong S$ is the proper transform of S. The linear system $|C_0|$ on $\bar{V}$ defines a $\mathbb{P}^1$-fibration $\Phi_0 : \bar{V} \to \mathbb{P}^1$ for which C_0 is a fiber and C_1 is a section. Choosing an appropriate affine coordinate on $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ we may suppose that $\Phi_0^{-1}(\infty) = C_0$ and $\Phi_0^{-1}(0)$ contains the subchain $C_2 + \ldots + C_n$ of D. The reduced curve $D_{\mathrm{ext}} = \Phi_0^{-1}(0) \cup C_0 \cup C_1$ is called the *extended divisor* of the completion $(\bar{V}, D)$ of V. The following lemma appeared implicitly in the proof of Proposition 1 in [Gi] (cf. also [FKZ_1]). To make this note self-contained we provide a short argument.

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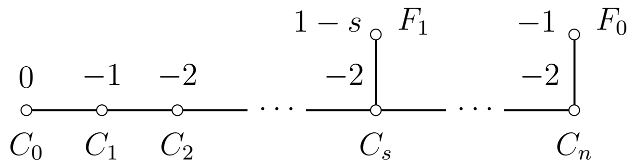
Lemma 2.1. *(a) For every $a \neq 0$ the fiber $\Phi_0^{-1}(a)$ is reduced and isomorphic to $\mathbb{P}^1$.*
(b) $D_{\mathrm{ext}} = \Phi_0^{-1}(0) \cup C_0 \cup C_1$ is an SNC divisor with dual graph

(1) D_{ext} :



for some s with $2 \leq s \leq n$.

Proof. (a) follows easily from the fact that the affine surface $V = \bar{V} \backslash D$ does not contain complete curves.

To deduce (b), we note first that $\bar{V}$ has Picard number $n+2$, since $\bar{V}$ is obtained from Σ_d by a sequence of n blowups. Since $C_1 \cdot C_2 = 1$, the part $\Phi_0^{-1}(0) - C_2$ of the fiber $\Phi_0^{-1}(0)$ can be blown down to a smooth point. Since $C_1^2 = -1$, after this contraction we arrive at the Hirzebruch surface Σ_1, which has Picard number 2. Hence the fiber $\Phi_0^{-1}(0)$ consists of $n+1$ components. In other words, $\Phi_0^{-1}(0)$ contains, besides the chain $C_2 + \ldots + C_n$, exactly 2 further components F_0 and F_1 called *feathers* [FKZ$_1$]. These are disjoint smooth rational curves, which meet the chain $C_2 + \ldots + C_n$ transversally at two distinct smooth points. Indeed, $\Phi_0^{-1}(0)$ is an SNC divisor without cycles and the affine surface V does not contain complete curves. In particular, $(F_0 \cup F_1) \setminus D$ is a union of two disjoint smooth curves on V isomorphic to $\mathbb{A}^1$.

Since $\Phi_0^{-1}(0) - C_2$ can be blown down to a smooth point and $C_i^2 = -2$ for $i \geq 2$, at least one of these feathers, call it F_0, must be a (-1)-curve. We claim that F_0 cannot meet a component C_r with $3 \leq r \leq n-1$. Indeed, otherwise the contraction of $F_0 + C_r + C_{r+1}$ would result in $C_{r-1}^2 = 0$ without the total fiber over 0 being irreducible, which is impossible. Hence F_0 meets either C_2 or C_n.

If F_0 meets C_2 then $F_0 + C_2 + \ldots + C_n$ is contractible to a smooth point. Thus the image of F_1 will become a smooth fiber of the contracted surface. This is only possible if F_1 is a (-1)-curve attached to C_n. Hence after interchanging F_0 and F_1 the divisor D_{ext} is as in (1) with $s = 2$.

Therefore we may assume for the rest of the proof that F_0 is attached at C_n and F_1 at C_s, where $2 \leq s \leq n$. Contracting the chain $F_0 + C_2 + \ldots + C_n$ within the fiber $\Phi_0^{-1}(0)$ yields an irreducible fiber F_1' with $(F_1')^2 = 0$. This determines the index s in a unique way, namely, $s = 1 - F_1^2$. □

2.2. **Jumping feathers.** The construction in 2.1 depends on the initial choice of the point $A \in S$. In particular, the extended divisor $D_{\mathrm{ext}} = D_{\mathrm{ext}}(A)$ and the integer $s = s(A)$ depend on A. The aim of this subsection is to show that $s(A) = 2$ for a general choice of $A \in S$.

2.2. Let $\bar{F}_0 = \bar{F}_0(A)$ and $\bar{F}_1 = \bar{F}_1(A)$ denote the images of the feathers $F_0 = F_0(A)$ and $F_1 = F_1(A)$, respectively, in the Hirzebruch surface Σ_d under the blowdown $\sigma : \bar{V} \to \Sigma_d$ of the chain $C_1 + \ldots + C_n$. These images meet each other and the original section $S = \sigma(C_0)$ at the point A and satisfy

(2) $$\bar{F}_0^2 = 0, \quad \bar{F}_0 \cdot \bar{F}_1 = \bar{F}_0 \cdot S = 1, \quad \bar{F}_1^2 = n - 2s + 2, \quad \bar{F}_1 \cdot S = n - s + 1\,,$$

where $s = s(A)$. Hence $\bar{F}_0 = \bar{F}_0(A)$ is the fiber through A of the canonical projection $\pi : \Sigma_d \to \mathbb{P}^1$ and $\bar{F}_1 = \bar{F}_1(A)$ is a section of π. The sections S and $\bar{F}_1$ meet only at A, where they can be tangent (osculating).

We let below

$$s_0 = s(A_0) = \min_{A\in S}\{s(A)\}\,, \quad l = \bar{F}_1(A_0)^2 + 1 \text{ and } m = \bar{F}_1(A_0)\cdot S\,. \tag{3}$$

For the next proposition see e.g., Lemma 7 and the following Remark in [Gi], or Proposition 4.8.11 in [DaGi, II]. Our proof is based essentially on the same idea.

Proposition 2.3. *(a) $s(A) = s_0$ for a general point $A \in S$, and*
(b) $s_0 = 2$.

Proof. For a general point $A \in S$ and an arbitrary point $A' \in S$ we have $\bar{F}_1(A) \sim \bar{F}_1(A') + k\bar{F}_0$ for some $k \geq 0$. Hence $\bar{F}_1(A)^2 = \bar{F}_1(A')^2 + 2k \geq \bar{F}_1(A')^2$. Using (2) it follows that

$$s(A) = 1 - F_1(A)^2 \leq s(A') = 1 - F_1(A')^2\,.$$

Thus $s(A) = s_0$ for all points A in a Zariski open subset $S_0 \subseteq S$, which implies (a).

To deduce (b) we note that by (3)

$$l = n - 2s_0 + 3 \leq n - s_0 + 1 = m$$

with equality if and only if $s_0 = 2$. Thus it is enough to show that $l \geq m$. Restriction to S yields

$$\bar{F}_1(A)|S = m[A] \in \mathrm{Div}(S) \qquad \forall A \in S_0\,. \tag{4}$$

Consider the linear systems

$$|\bar{F}_1(A_0)| \cong \mathbb{P}^l \quad \text{and} \quad |\mathcal{O}_S(m)| \cong \mathbb{P}^m$$

on Σ_d and $S \cong \mathbb{P}^1$, respectively, and the linear map

$$\rho : \mathbb{P}^l \dashrightarrow \mathbb{P}^m, \quad F \longmapsto F|S\,.$$

The set of divisors

$$\Gamma_m = \{m[A]\}_{A\in S}$$

represents a rational normal curve of degree m in $\mathbb{P}^m = |\mathcal{O}_S(m)|$. In view of (4) the linear subspace $\overline{\rho(\mathbb{P}^l)}$ contains Γ_m. Since the curve Γ_m is linearly non-degenerate we have $\overline{\rho(\mathbb{P}^l)} = \mathbb{P}^m$and so $l \geq m$, as desired. □

2.3. **Elementary shifts.** We consider as before a completion $V = \bar{V} \setminus D$ of a Danilov-Gizatullin surface V as in 2.1.

2.4. Choosing A generically, according to Proposition 2.3 we may suppose in the sequel that $s = s(A) = 2$ and $F_0^2 = F_1^2 = -1$.

By (1) in Lemma 2.1, blowing down in $\bar{V}$ the feathers F_0, F_1 and then the chain $C_3 + \ldots + C_n$ yields the Hirzebruch surface Σ_1, in which C_0 and C_2 become fibers and C_1 a section. Reversing this contraction, the above completion $\bar{V}$ can be obtained from Σ_1 by a sequence of blowups as follows. The sequence starts by the blowup with center at a point $P_3 \in C_2\backslash C_1$ to create the next component C_3 of the zigzag D. Then we perform subsequent blowups with centers at points $P_4, \ldots, P_{n+1}$ infinitesimally near to P_3, where for each $i = 4, \ldots, n$ the blowup of $P_i \in C_{i-1}\backslash C_{i-2}$ creates the next component C_i of the zigzag. The blowup with center at $P_{n+1} \in C_n\backslash C_{n-1}$ creates the feather F_0. Finally we blow up at a point $Q \in C_2\backslash C_1$ different from P_3 to create the feather F_1. In this way we recover the given completion $\bar{V}$ with extended divisor D_{ext} as in (1), where $s = 2$.

We observe that the sequence $P_3,\ldots,P_{n+1},Q$ depends on the original triplet (Σ_d,S,A). It follows that, varying the points $P_3,\ldots,P_{n+1},Q$ and then contracting the chain $C_1+\ldots+C_n=D-C_0$ on the resulting surface $\bar V$, we can obtain all possible Danilov-Gizatullin surfaces

$$V=\bar V\setminus D\cong\Sigma_d\setminus S\quad\text{with}\quad S^2=n\ \text{ and }\ 0\le d\le n-1\,.$$

Thus to deduce the Danilov-Gizatullin Isomorphism Theorem 1.1 it suffices to establish the following fact.

Proposition 2.5. *The isomorphism type of the affine surface $V=\bar V\backslash D$ does not depend on the choice of the blowup centers $P_3,\ldots,P_{n+1}$ and Q as above.*

The proof proceeds in several steps.

2.6. First we note that in our construction it suffices to keep track only of some partial completions rather than of the whole complete surfaces. We can choose affine coordinates (x,y) in $\Sigma_1\backslash(C_0\cup C_1)\cong\mathbb{A}^2$ so that $C_2\backslash C_1=\{x=0\}$, $P=P_3=(0,0)$ and $Q=(0,1)$. The affine surface V can be obtained from the affine plane $\mathbb{A}^2$ by performing subsequent blowups with centers at the points $P_3,\ldots,P_{n+1}$ and Q as in 2.4 and then deleting the curve $C_2\cup\ldots\cup C_n=D\backslash(C_0\cup C_1)$.

With $X_2=\mathbb{A}^2$, for every $i=3,\ldots,n+1$ we let X_i denote the result of the subsequent blowups of $\mathbb{A}^2$ with centers $P_3,\ldots,P_i$. This gives a tower of blowups

$$\bar V\backslash(C_0\cup C_1)=:X_{n+2}\to X_{n+1}\to X_n\to\ldots\to X_2=\mathbb{A}^2\,,\tag{5}$$

where in the last step the point Q is blown up to create F_1.

2.7. Let us exhibit a special case of this construction. Consider the standard action

$$(\lambda_1,\lambda_2):(x,y)\mapsto(\lambda_1x,\lambda_2y)$$

of the 2-torus $\mathbb{T}=(\mathbb{C}^*)^2$ on the affine plane $X_2=\mathbb{A}^2$. We claim that there is a unique sequence of points $(0,0)=P_3=P_3^o,\ldots,P_{n+1}=P_{n+1}^o$ such that the torus action can be lifted to X_i for $i=3,\ldots,n+1$. Indeed, if by induction the $\mathbb{T}$-action is lifted already to X_i with $i\ge2$, then on $C_i\backslash C_{i-1}\cong\mathbb{A}^1$ the induced $\mathbb{T}$-action has a unique fixed point P_{i+1}^o. Blowing up this point the $\mathbb{T}$-action can be lifted further to X_{i+1}. Blowing up finally $Q=(0,1)\in C_2\setminus C_1$ and deleting $C_2\cup\ldots\cup C_n$ we arrive at a unique *standard* Danilov-Gizatullin surface $V_{\mathrm{st}}=V_{\mathrm{st}}(n)$.

Let us note that $\mathbb{T}$ acts transitively on $(C_2\setminus C_1)\setminus\{(0,0)\}$. Thus up to isomorphism, the resulting affine surface V_{st} does not depend on the choice of Q.

2.8. Consider now an automorphism h of $\mathbb{A}^2$ fixing the y-axis pointwise. It moves the blowup centers $P_4,\ldots,P_{n+1}$ to new positions $P_4',\ldots,P_{n+1}'$, while P_3 and Q remain unchanged. It is easily seen that h induces an isomorphism between V and the resulting new affine surface V'. We show in Lemma 2.9 below that applying a suitable automorphism h, we can choose V' to be the standard surface V_{st} as in 2.7. This implies immediately Proposition 2.5 and as well Theorem 1.1. More precisely, our h will be composed of *elementary shifts*

$$h_{a,t}:(x,y)\mapsto(x,y+ax^t),\quad\text{where}\quad a\in\mathbb{C}\quad\text{and}\quad t\ge0\,.\tag{6}$$

Lemma 2.9. *By a sequence of elementary shifts as in (6) we can move the blowup centers $P_4,\ldots,P_n$ into the points $P_4^o,\ldots,P_n^o$ so that V is isomorphic to V_{st}.*

Proof. Since $X_2 = \mathbb{A}^2$ the assertion is obviously true for $i = 2$. The point $P_3 = (0,0)$ being fixed by $\mathbb{T}$, the torus action can be lifted to X_3. The blowup with center at P_3 has a coordinate presentation

$$(x_3, y_3) = (x, y/x)\,, \quad \text{or, equivalently,} \quad (x, y) = (x_3, x_3y_3)\,,$$

where the exceptional curve C_3 is given by $x_3 = 0$ and the proper transform of C_2 by $y_3 = \infty$. The action of $\mathbb{T}$ in these coordinates is

$$(\lambda_1, \lambda_2).(x_3, y_3) = (\lambda_1 x_3, \lambda_1^{-1}\lambda_2 y_3)\,,$$

while the elementary shift $h_{a,t}$ can be written as

$$h_{a,t} : (x_3, y_3) \mapsto (x_3, y_3 + ax_3^{t-1})\,. \tag{7}$$

Thus in (x_3, y_3)-coordinates $P_4^o = (0,0)$. Furthermore for $t = 1$, the shift $h_{a,1}$ yields a translation on the axis $C_3 \backslash C_2 = \{x_3 = 0\}$, while $h_{a,t}$ with $t \geq 2$ is the identity on this axis. Applying $h_{a,1}$ for a suitable a we can move the point $P_4 \in C_3 \backslash C_2$ to P_4^o. Repeating the argument recursively, we can achieve that $P_i = P_i^o$ for $i \leq n+1$, as required. □

Remarks 2.10. 1. The surface X_{n+1} as in 2.7 is toric, and the $\mathbb{T}$-action on X_{n+1} stabilizes the chain $C_2 \cup \ldots \cup C_n \cup F_0$. There is a 1-parameter subgroup G of the torus (namely, the stationary subgroup of the point $Q = (0,1)$), which lifts to X_{n+2} and then restricts to $V_{\mathrm{st}} = X_{n+2} \setminus (C_2 \cup \ldots \cup C_n)$. Fixing an isomorphism $G \cong \mathbb{C}^*$ gives a $\mathbb{C}^*$-action on V_{st}. As follows from [FKZ$_2$, 1.0.6], $V_{\mathrm{st}} = V_{\mathrm{st}}(n)$ is the normalization of the surface $W_n \subseteq \mathbb{A}^3$ with equation

$$x^{n-1}y = (z-1)(z+1)^{n-1}\,.$$

For $n \geq 3$ this surface has non-isolated singularities, and is equipped with the $\mathbb{C}^*$-action $\lambda.(x, y, z) = (\lambda x, \lambda^{n-1}y, z)$. Due to the Danilov-Gizatullin Isomorphism Theorem 1.1, any Danilov-Gizatullin surface V_n is isomorphic to the normalization of W_n.

2. However, the specific $\mathbb{C}^*$-action on V_n obtained in this way is not unique as was observed by Peter Russell. According to Proposition 5.14 in [FKZ$_1$], in $\mathrm{Aut}(V_n)$ there are exactly $n-1$ different conjugacy classes of such actions corresponding to different choices of $s = 2, \ldots, n$ in diagram (1). Let us sketch a construction of these classes which does not rely on DPD-presentations as in *loc.cit*, but follows a procedure similar to those used in the proof above.

Given $s \in \{2, \ldots, n\}$, starting with $\bar{X}_2 = \Sigma_1 \to \mathbb{P}^1$ and a chain $C_0 + C_1 + C_2$ on Σ_1 as in 2.4 and 2.6, we blow up the point $(0,0) \in C_2$ creating the feather F_1, then at the point $C_2 \cap F_1$ creating C_3 etc., until the component C_s is created. The standard torus action on Σ_1 lifts to the resulting surface $\bar{X}_{s+1}$ stabilizing the linear chain $F_1 + C_0 + \ldots + C_s$. Next we blowup at a point $P \in C_s \backslash (F_1 \cup C_{s-1})$ creating a new component C_{s+1}, and we lift the action of the 1-parameter subgroup $G = \mathrm{Stab}_P(\mathbb{T})$ to the resulting surface $\bar{X}_{s+2}$. Choosing an appropriate isomorphism $G \cong \mathbb{C}^*$ we may assume that C_s is attractive for the resulting $\mathbb{C}^*$-action Λ_s on $\bar{X}_{s+2}$. We continue blowing up subsequently at the fixed points of this action on the curves $C_{i+1} \setminus C_i$, $i = s, \ldots, n$ creating components $C_{s+2}, \ldots, C_n$ and the feather F_0. Finally we arrive at a $\mathbb{C}^*$-surface $\bar{V} = \bar{X}_{n+2}$ with an extended divisor as in (1). Contracting $C_1 + \ldots + C_n$ exhibits the open part $V = \bar{V} \setminus D$, where $D = C_0 + \ldots + C_n$, as a complement to an ample section in a Hirzebruch surface. Thus $V = V_n$ is a Danilov-Gizatullin surface of index n endowed with a $\mathbb{C}^*$-action say,

Λ_s, such that $\bar{V}$ is its equivariant standard completion. Note that the isomorphism class of $(\bar{V}, D)$ is independent on the choice of the point $P \in C_s \setminus (F_1 \cup C_{s-1})$. Indeed this point can be moved by the $\mathbb{T}$-action yielding conjugated $\mathbb{C}^*$-actions on V_n.

Contracting the chain $C_1 + \ldots + C_n$ leads to a Hirzebruch surface Σ_d such that the image of F_0 is a fiber of the ruling $\Sigma_d \to \mathbb{P}^1$. Moreover, the image S of C_0 is an ample section with $S^2 = n$ so that $V_n = \Sigma_d \setminus S$. The image of F_1 is another section with $F_1^2 = n + 2 - 2s$. In particular, if this number is negative then $d = 2s - 2 - n$.

One can show that the Λ_s, $s = 2, \ldots, n$ represent all conjugacy classes of $\mathbb{C}^*$-actions on V_n. Moreover, inverting the action Λ_s with respect to the isomorphism $t \mapsto t^{-1}$ of $\mathbb{C}^*$ yields the action Λ_{n-s+2}. Thus after inversion, if necessary, we may suppose that $2s - 2 \geq n$ so that $V_n \cong \Sigma_d \backslash S$ as above with $d = 2s - 2 - n$.

3. As was remarked by Peter Russell, with the exception of Proposition 2.3 our proof is also valid for Danilov-Gizatullin surfaces over an algebraically closed field of any characteristic p. Moreover Proposition 2.3 holds as soon as $p = 0$ or p and m are coprime. In particular it follows that the Isomorphism Theorem holds in the cases $p = 0$ and $p \geq n - 2$. This latter result was shown already in [DaGi]. However for $p = 2$ and $n = 56$ there is an infinite number of isomorphism types of Danilov-Gizatullin surfaces; see [DaGi, §9].

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